THREE MOMENT METHOD

Continuous Beams - Clapeyron's "three-moment" equation

Introduction

1. Developed by French Engineer Clapeyron in 1857.
2. This equation relates the internal moments in a continuous beam at three points of support to the loads acting between the supports.
3. By successive application of this equation to each span of the beam, one obtains a set of equations that may be solved simultaneously for the unknown internal moments at the support.

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Move one span further and repeat the procedure.

5. In a 3 span beam, the mid-moment from step 3 above (B), could now be solved using the two equations from step 4 and 3 together, by writing 2 equations with 2unknowns.
6. Repeat as needed, always moving one span to the right and writing a new set of moment equations.
7. Solve 2 simultaneous equations for 3 spans, or 3 equations for more than 3 spans, to get the interior moments.
8. Once all interior moments are known, solve for reactions using free body diagrams of individual spans.
9. Draw shear and moment diagrams as usual. This will also serve as a check for the moment values.

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A continuous beam is a statically indeterminate multispan beam on hinged support. The end spans may be cantilever, may be freely supported or fixed supported. At least one of the supports of a continuous beam must be able to develop a reaction along the beam axis.

- Beams are made continuous over the supports to increase structural integrity.
- A continuous beam provides an alternate load path in the case of failure at a section.
- In regions with high seismic risk, continuous beams and frames are preferred in buildings and bridges.
- A continuous beam is a statically indeterminate structure.

Continuous Beam - Clapeyron's "Three-Moment" equation

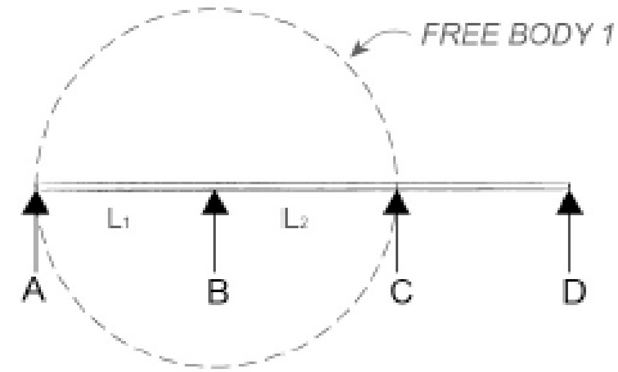
- When a beam is supported on more than two supports it is termed continuous.
- In cases such as these it is not possible to determine directly the reactions at the three supports by the normal equations of static equilibrium since there are too many unknowns.
- An extension of Mohr's area-moment method is therefore used to obtain a relationship between the B.M.s at the supports, from which the reaction values can then be determined and the B.M. and S.F. diagrams drawn.

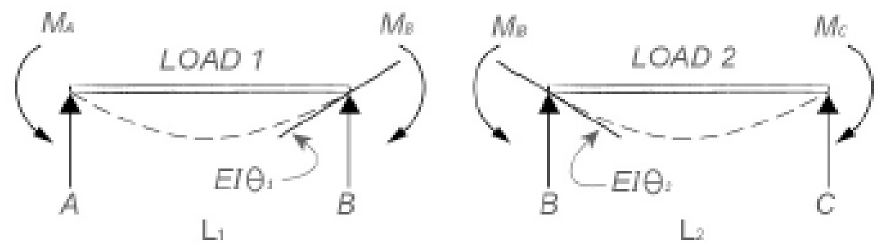
Continuous Beams - Clapeyron's "three-moment" equation

- Any number of spans
- Symmetric or non-symmetric

<u>Procedure:</u>

1. Draw a free body diagram of the first two spans.
2. Label the spans L1 and L2 and the supports (or free end) A, B and C as show.
3. Use the Three-Moment equation to solve for each unknown moment, either as a value or as an equation.





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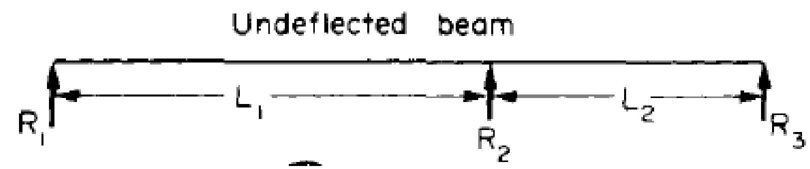
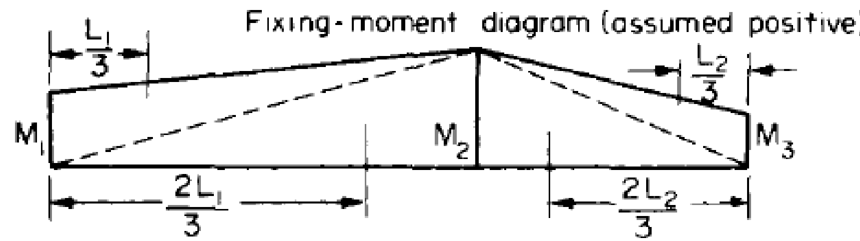
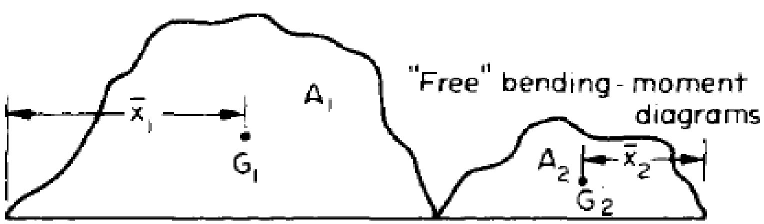
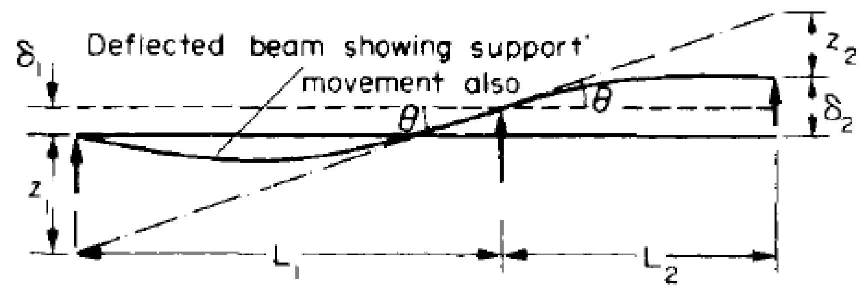
Consider therefore the beam shown

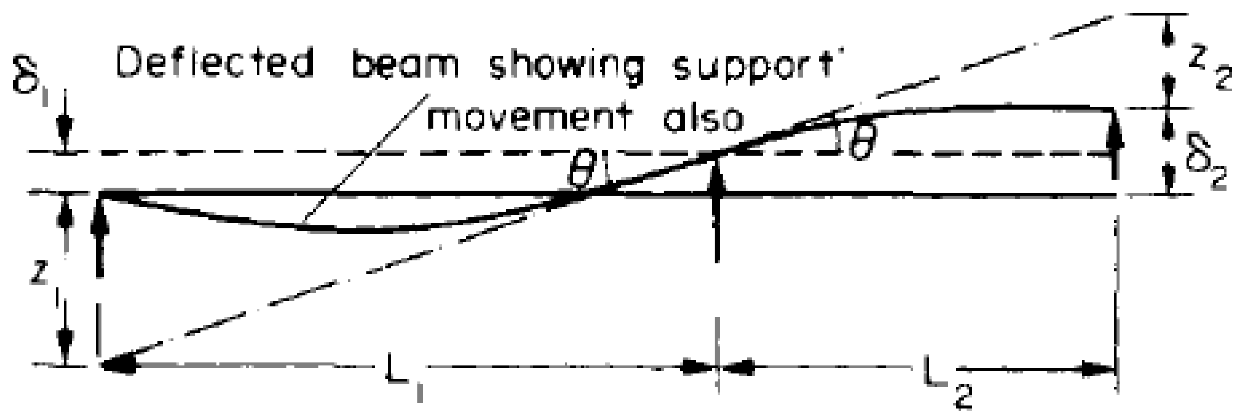








Continuous Beam - Clapeyron's "Three-Moment" equation



The bottom figure shows the deflected position of the beam, the deflections $\boldsymbol{\delta_1}$ and $\boldsymbol{\delta_2}$ being relative to the left-hand support.

If a tangent is drawn at the centre support then the intercepts at the end of each span are $\boldsymbol{z_1}$ and $\boldsymbol{z_2}$ and $\boldsymbol{\theta}$ is the slope of the tangent, and hence the beam, at the centre support.

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Now, assuming deflections are small,

$$\theta \text{ (radians)} = \frac{z_1 + \delta_1}{L_1} = \frac{z_2 + \delta_2 - \delta_1}{L_2}$$

$$\therefore \quad \frac{z_1}{L_1} + \frac{\delta_1}{L_1} = \frac{z_2}{L_2} + \frac{(\delta_2 - \delta_1)}{L_2}$$

But from Mohr's area–moment method,

$$z = \frac{A\bar{x}}{EI}$$

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where A is the area of the B.M. diagram over the span to which z refers.

$$\therefore \quad z_1 = -\frac{1}{EI_1}\left[A_1\bar{x}_1 + \left(\frac{M_1L_1}{2}\times\frac{L_1}{3}\right) + \left(\frac{M_2L_1}{2}\times\frac{2L_1}{3}\right)\right]$$

$$= -\frac{1}{EI_1}\left[A_1\bar{x}_1 + \frac{M_1L_1^2}{6} + \frac{M_2L_1^2}{3}\right]$$

and

$$z_2 = \frac{1}{EI_2}\left[A_2\bar{x}_2 + \left(\frac{M_3L_2}{2}\times\frac{L_2}{3}\right) + \left(\frac{M_2L_2}{2}\times\frac{2L_2}{3}\right)\right]$$

$$= \frac{1}{EI_2}\left[A_2\bar{x}_2 + \frac{M_3L_2^2}{6} + \frac{M_2L_2^2}{3}\right]$$

Since the intercepts are in opposite directions, they are of opposite sign.

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$$\therefore \quad -\frac{\left[A_1\bar{x}_1 + \frac{M_1 L_1^2}{6} + \frac{M_2 L_1^2}{3}\right]}{EI_1 L_1} + \frac{\delta_1}{L_1} = \frac{\left[A_2\bar{x}_2 + \frac{M_3 L_2^2}{6} + \frac{M_2 L_2^2}{3}\right]}{EI_2 L_2} + \frac{(\delta_2 - \delta_1)}{L_2}$$

$$\therefore \quad -\frac{A_1\bar{x}_1}{I_1 L_1} - \frac{M_1 L_1}{6I_1} - \frac{M_2 L_1}{3I_1} + \frac{E\delta_1}{L_1} = \frac{A_2\bar{x}_2}{I_2 L_2} + \frac{M_3 L_2}{6I_2} + \frac{M_2 L_2}{3I_2} + \frac{E(\delta_2 - \delta_1)}{L_2}$$

$$\therefore \quad -\frac{M_1 L_1}{I_1} - 2M_2\left[\frac{L_1}{I_1} + \frac{L_2}{I_2}\right] - \frac{M_3 L_2}{I_2} = 6\left[\frac{A_1\bar{x}_1}{I_1 L_1} + \frac{A_2\bar{x}_2}{I_2 L_2}\right] + 6E\left[\frac{(\delta_2 - \delta_1)}{L_2} - \frac{\delta_1}{L_1}\right]$$

This is the full three-moment equation; it can be greatly simplified **if the beam is uniform**, i.e. $I_1 = I_2 = I$, as follows:

$$-M_1 L_1 - 2M_2[L_1 + L_2] - M_3 L_2 = 6\left[\frac{A_1\bar{x}_1}{L_1} + \frac{A_2\bar{x}_2}{L_2}\right] + 6EI\left[\frac{(\delta_2 - \delta_1)}{L_2} - \frac{\delta_1}{L_1}\right]$$

If the supports are on the same level, i.e. $\delta_1 = \delta_2 = 0$,

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$$\therefore \quad -\frac{\left[A_1\bar{x}_1 + \frac{M_1 L_1^2}{6} + \frac{M_2 L_1^2}{3}\right]}{EI_1 L_1} + \frac{\delta_1}{L_1} = \frac{\left[A_2\bar{x}_2 + \frac{M_3 L_2^2}{6} + \frac{M_2 L_2^2}{3}\right]}{EI_2 L_2} + \frac{(\delta_2 - \delta_1)}{L_2}$$

$$\therefore \quad -\frac{A_1\bar{x}_1}{I_1 L_1} - \frac{M_1 L_1}{6I_1} - \frac{M_2 L_1}{3I_1} + \frac{E\delta_1}{L_1} = \frac{A_2\bar{x}_2}{I_2 L_2} + \frac{M_3 L_2}{6I_2} + \frac{M_2 L_2}{3I_2} + \frac{E(\delta_2 - \delta_1)}{L_2}$$

$$\therefore \quad -\frac{M_1 L_1}{I_1} - 2M_2\left[\frac{L_1}{I_1} + \frac{L_2}{I_2}\right] - \frac{M_3 L_2}{I_2} = 6\left[\frac{A_1\bar{x}_1}{I_1 L_1} + \frac{A_2\bar{x}_2}{I_2 L_2}\right] + 6E\left[\frac{(\delta_2 - \delta_1)}{L_2} - \frac{\delta_1}{L_1}\right]$$

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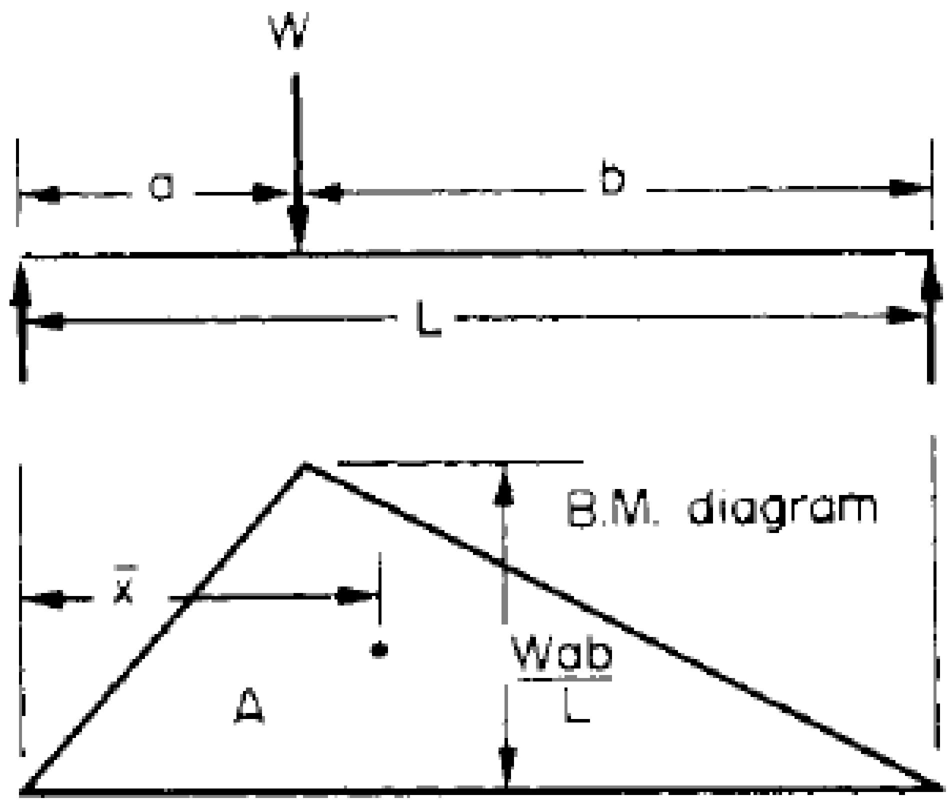
$$\boxed{-M_1 L_1 - 2M_2[L_1 + L_2] - M_3 L_2 = 6\left[\frac{A_1\bar{x}_1}{L_1} + \frac{A_2\bar{x}_2}{L_2}\right]}$$

This is the form in which Clapeyron's three-moment equation is normally used.

Continuous Beam - Clapeyron's "Three-Moment" equation

The following standard results for $\frac{6A\bar{x}}{L}$ are very useful:

(1) *Concentrated loads*



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$$\frac{6A\bar{x}}{L} = \frac{6}{L}\left[\frac{Wab}{L} \times \frac{a}{2} \times \frac{2a}{3} + \frac{Wab}{L} \times \frac{b}{2}\left(a + \frac{b}{3}\right)\right]$$

$$= \frac{6Wab}{L^2}\left[\frac{a^2}{3} + \frac{ab}{2} + \frac{b^2}{6}\right]$$

$$= \frac{Wab}{L^2}[2a^2 + 3ab + b^2] = \frac{Wab}{L^2}(2a + b)(a + b)$$

$$= \frac{Wab}{L}(2a + b)$$

But $\quad b = L - a$

$\therefore$

$$\frac{6A\bar{x}}{L} = \frac{Wa}{L}(L - a)(2a + L - a)$$

$$= \frac{Wa}{L}(L^2 - a^2)$$

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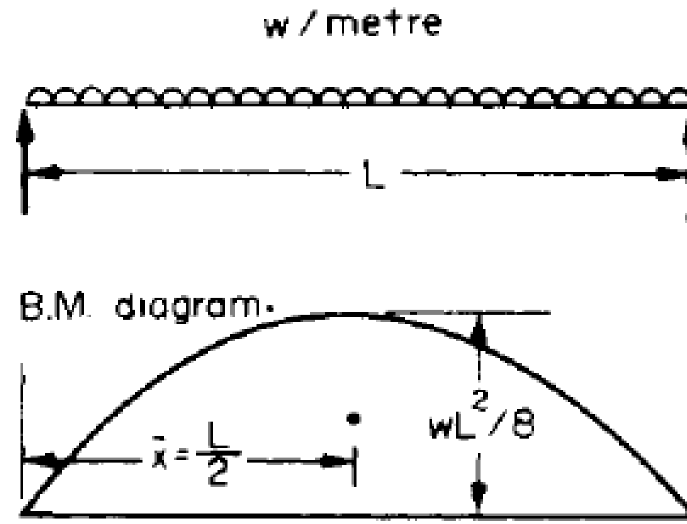
(2) *Uniformly distributed loads*

Here the B.M. diagram is a parabola for which

$$\text{area} = \tfrac{2}{3}\ \text{base} \times \text{height}$$

$$\therefore \qquad \frac{6A\bar{x}}{L} = \frac{6}{L} \times \frac{2}{3} \times L \times \frac{wL^2}{8} \times \frac{L}{2}$$

$$= \frac{wL^3}{4}$$



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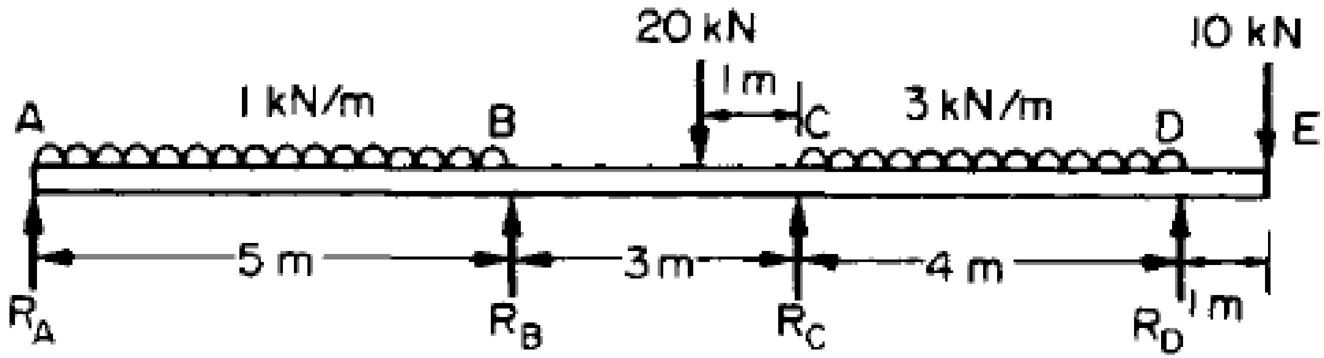
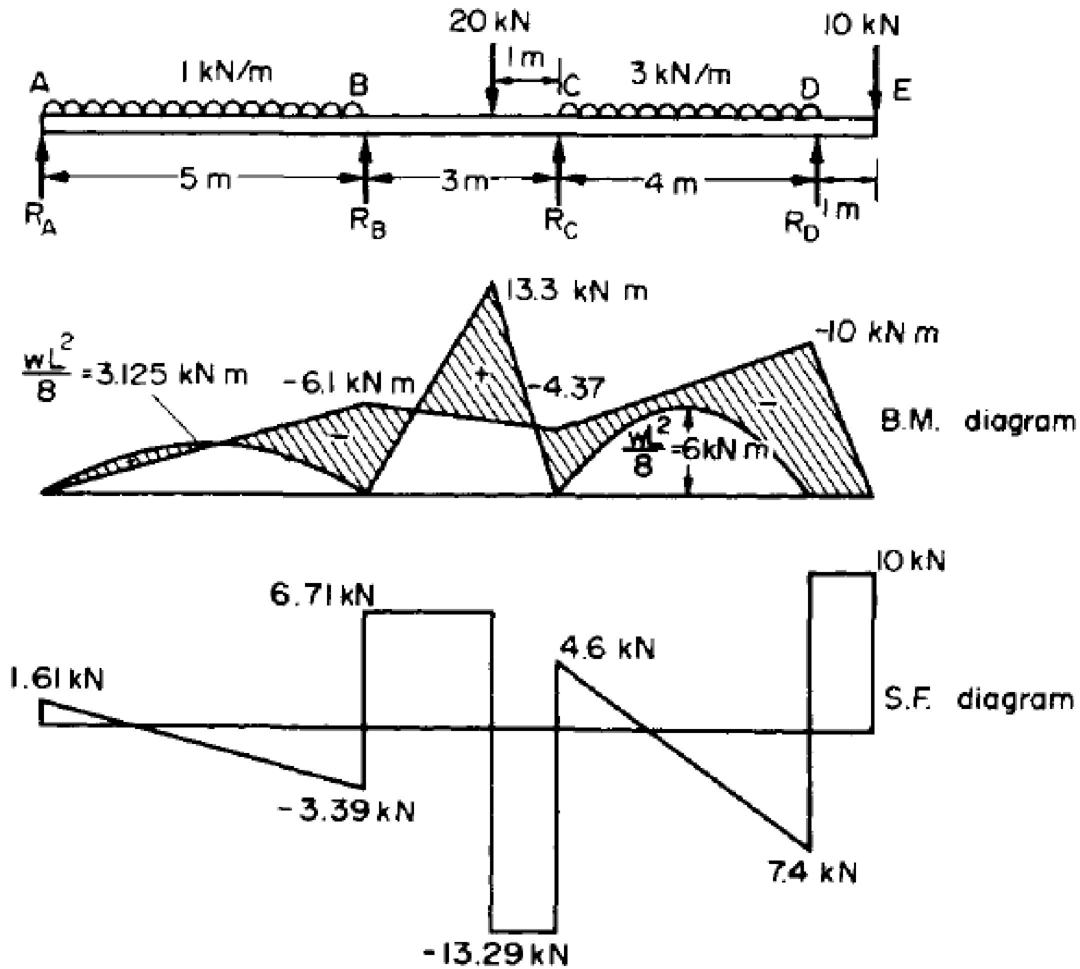
Example 5.7

A beam *ABCDE* is continuous over four supports and carries the loads shown in Fig. 5.43. Determine the values of the fixing moment at each support and hence draw the S.F. and B.M. diagrams for the beam.



Continuous Beam - Clapeyron's "Three-Moment" equation



Continuous Beam - Clapeyron's "Three-Moment" equation

Solution

By inspection, $M_A = 0$ and $M_D = -1 \times 10 = -10\,\text{kN m}$

Applying the three-moment equation for the first two spans,

$$-M_A L_1 - 2M_B(L_1 + L_2) - M_C L_2 = \frac{wL_1^3}{4} + \frac{wa}{L_2}(L_2^2 - a^2)$$

$$0 - 2M_B(5+3) - 3M_C = \left[\frac{1 \times 5^3}{4} + \frac{20 \times 1}{3}(3^2 - 1^2)\right]10^3$$

$$-16M_B - 3M_C = (31.25 + 53.33)10^3$$

$$-16M_B - 3M_C = 84.58 \times 10^3$$

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Solution

and, for the second and third spans,

$$-M_BL_2 - 2M_C(L_2 + L_3) - M_DL_3 = \frac{wa}{L_2}(L_2^2 - a^2) + \frac{wL_3^3}{4}$$

$$-3M_B - 2M_C(3+4) - (-10 \times 10^3)4 = \left[\frac{20 \times 2}{3}(3^2 - 2^2) + \frac{(3 \times 4^3)}{4}\right]10^3$$

$$-3M_B - 14M_C + (40 \times 10^3) = (66.67 + 48)10^3$$

$$-3M_B - 14M_C = 74.67 \times 10^3$$

(2) × 16/3 $$-16M_B - 74.67M_C = 398.24 \times 10^3$$

(3) − (1) $$-71.67M_C = 313.66 \times 10^3$$

$$M_C = \mathbf{-4.37 \times 10^3\ Nm}$$

Continuous Beam - Clapeyron's "Three-Moment" equation

Solution

Substituting in (1),

$$-16M_B - 3(-4.37 \times 10^3) = 84.58 \times 10^3$$

$$M_B = -\frac{(84.58 - 13.11)10^3}{16}$$

$$= -\mathbf{4.47\ kN\ m}$$

Moments about B (to left),

$$R_A \times 5 - \left(\frac{1 \times 10^3}{2} \times 5^2\right) = -4.47 \times 10^3$$

$$5R_A = (-4.47 + 12.5)10^3$$

$$R_A = \mathbf{1.61\ kN}$$

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Solution

Then, since

$$R_A + R_B + R_C + R_D = 47\,\text{kN}$$
$$1.61 + 10.1 + R_C + 17.4 = 47$$
$$R_C = \mathbf{17.9\,kN}$$

This value should then be checked by taking moments to the right of B,

$$(-10 \times 10^3 \times 8) + 7R_D + 3R_C - (3 \times 10^3 \times 4 \times 5) - (20 \times 10^3 \times 2) = -4.47 \times 10^3$$
$$3R_C = (-4.47 + 40 + 60 + 80 - 121.8)10^3 = 53.73 \times 10^3$$
$$R_C = \mathbf{17.9\,kN}$$