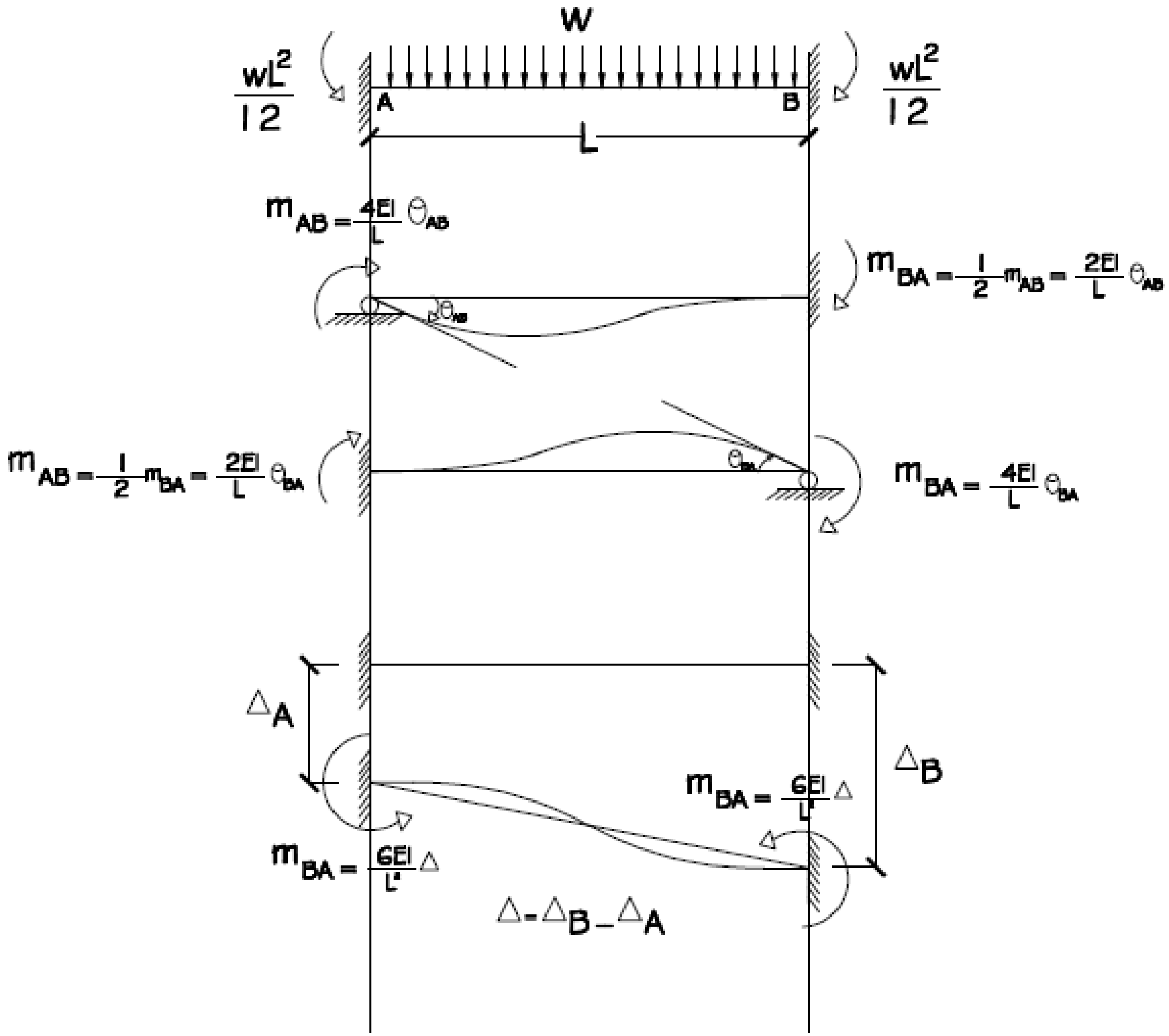
Slope Deflection Method for the Analysis of Indeterminate Structures

All structures must satisfy:

- **Load-displacement relationship**
- **Equilibrium of forces**
- **Compatibility of displacements**

Using the principle of superposition by considering separately the moments developed at each support of a typical prismatic beam (AB) shown in Fig. 1(a) of a continuous beam, due to each of the displacements , θ_A, θ_B, and Δ, , and the applied loads. Assume clockwise moments are +ive.

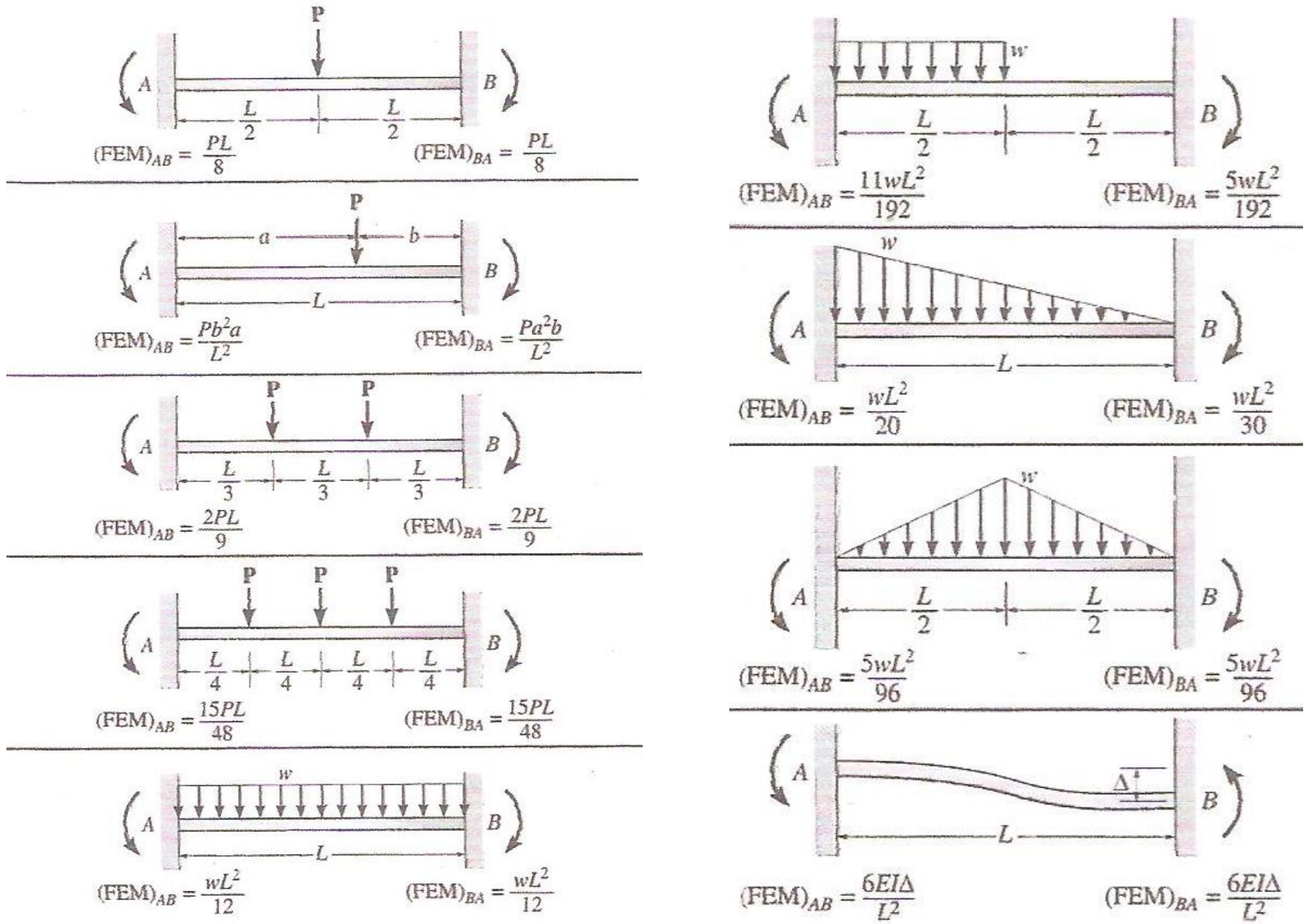


w
wL²/12
wL²/12
A
B
L
m_AB = 4EI/L θ_AB
m_BA = 1/2 m_AB = 2EI/L θ_AB
θ_AB
m_AB = 1/2 m_BA = 2EI/L θ_BA
θ_BA
m_BA = 4EI/L θ_BA
△A
△B
m_BA = 6EI/L² △
m_BA = 6EI/L² △
△=△B−△A

Fig (1)

1. **Assume ends A and B are fixed, i., e., the rotations $\theta_{AB} = \theta_{BA} = 0$. This means that we have to apply counterclockwise moment at end A $(FEM)_{AB}$ and clockwise moment at end B $(FEM)_{BA}$ due to the applied loads to cause zero rotation at each of ends A and B. Table (1) gives FEM for different loading conditions.**

Table (1)


P
A
B
L/2
L/2
(FEM)AB = PL/8
(FEM)BA = PL/8
P
a
b
L
(FEM)AB = Pb²a/L²
(FEM)BA = Pa²b/L²
P
P
L/3
L/3
L/3
(FEM)AB = 2PL/9
(FEM)BA = 2PL/9
P
P
P
L/4
L/4
L/4
L/4
(FEM)AB = 15PL/48
(FEM)BA = 15PL/48
w
L
(FEM)AB = wL²/12
(FEM)BA = wL²/12
w
L/2
L/2
(FEM)AB = 11wL²/192
(FEM)BA = 5wL²/192
w
L
(FEM)AB = wL²/20
(FEM)BA = wL²/30
w
L/2
L/2
(FEM)AB = 5wL²/96
(FEM)BA = 5wL²/96
Δ
L
(FEM)AB = 6EIΔ/L²
(FEM)BA = 6EIΔ/L²

2. Release end A against rotation at end A (rotates to its final position θ_{AB}) by applying clockwise moment m_{AB} while far end node B is held fixed as shown in Fig. 1.

3. Now, the clockwise moment m_{AB} - rotation θ_{AB} relationship is:

$$m_{AB} = \frac{4EI}{L}\theta_{AB}$$

4. The carry over moment at end B is:

$$m_{BA} = \frac{2EI}{L}\theta_{AB}$$

5. In a similar manner, $m_{BA} = \frac{4EI}{L}\theta_{BA}$ of the beam rotates to its final position θ_{BA}, while end A is held fixed. The clockwise moment m_{BA} – rotation θ_{BA}, relationship is:

$$m_{BA} = \frac{4EI}{L}\theta_{BA}$$

6. **The carry over moment at end A is:**

$$m_{AB} = \frac{2EI}{L}\theta_{BA}$$

7. **If node B is displaced relative to as shown in Fig. (1), so that the cord of the member rotates clockwise i., e., positive displacement and yet both ends do not rotate, then equal but anticlockwise moments are developed in the member as shown in the figure.**

$$m_{AB} = m_{BA} = -\frac{6EI}{l^2}\Delta$$

$$Where,$$

$$\Delta = \Delta_B - \Delta_A$$

Slope-Deflection Equation

Load-displacement relationship

If the end moments due to each displacement and the loading are added together, the resultant moments at the ends may then be written as:

$$M_{AB} = \frac{4EI}{l}\theta_{AB} + \frac{2EI}{l}\theta_{BA} - \frac{6EI}{l^2}\Delta - (FEM)_{AB} \qquad 1(a)$$

$$M_{BA} = \frac{2EI}{l}\theta_{AB} + \frac{4EI}{l}\theta_{BA} - \frac{6EI}{l^2}\Delta + (FEM)_{BA} \qquad 1(b)$$

.

For prismatic beam element, equation (1) may be written as:

$$M_{AB} = \frac{2EI}{l}(2\theta_{AB} + \theta_{BA} - 3\psi) - (FEM)_{AB} \qquad 2(a)$$

$$M_{BA} = \frac{2EI}{l}(\theta_{AB} + 2\theta_{BA} - 3\psi) + (FEM)_{BA} \qquad 2(b)$$

In which,

$$\psi = \frac{\Delta}{l}$$

The slope deflection equations (1 or 2) relate the unknown moments applied to the nodes to the displacements of the nodes for

<u>any span of the structure</u>.

To summarize application of the slope-deflection equations, consider the continuous beam shown in Fig. (2) which has four degrees of freedom.

Now equation (2) can be applied to each of the three spans.

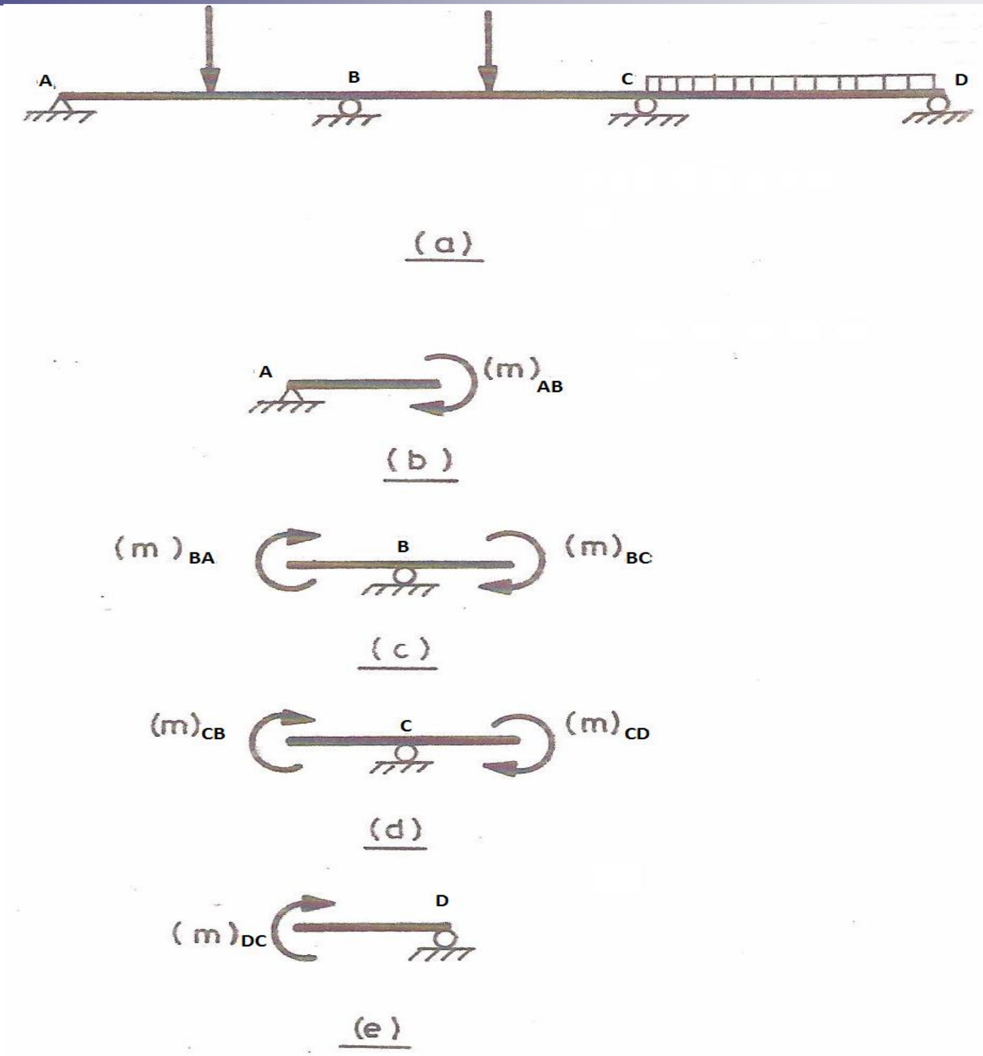


Fig. (2)

$$M_{AB} = (\frac{2EI}{l})_{AB}(2\theta_A + \theta_B) = 0$$

$$M_{BA} = (\frac{2EI}{l})_{AB}(\theta_A + 2\theta_B) + (FEM)_{BA}$$

$$M_{BC} = (\frac{2EI}{l})_{BC}(2\theta_B + \theta_C) - (FEM)_{BC}$$

$$M_{CB} = (\frac{2EI}{l})_{BC}(2\theta_C + \theta_B) + (FEM)_{CB}$$

$$M_{CD} = (\frac{2EI}{L})_{CD}(2\theta_C + \theta_D) - (FEM)_{CD}$$

$$M_{DC} = (\frac{2EI}{l})_{CD}(2\theta_D + \theta_C) = 0$$

From Fig.(2):

Equilibrium conditions

$$Joint(a); M_{AB} = 0$$

$$Joint(b); M_{BA} + M_{BC} = 0$$

$$Joint(c); M_{CB} + M_{CD} = 0$$

$$Joint(d); M_{DC} = 0$$

Compatibility conditions

$$Joint(a); \theta_{AB} = \theta_{A}$$

$$Joint(b); \theta_{BA} = \theta_{BC} = \theta_{B}$$

$$Joint(c); \theta_{CB} = \theta_{CD} = \theta_{C}$$

$$Joint(d); \theta_{DC} = \theta_{D}$$

$$Jo\,\text{int}(A):$$

$$(\frac{2EI}{l})_{AB}(2\theta_A+\theta_B)=0 \qquad (i)$$

$$Jo\,\text{int}(B):$$

$$(\frac{2EI}{l})_{AB}(\theta_A+2\theta_B)+(FEM)_{BA}+(\frac{2EI}{l})_{BC}(2\theta_B+\theta_C)-(FEM)_{BC}=0 \qquad (j)$$

$$Jo\,\text{int}(C):$$

$$(\frac{2EI}{l})_{BC}(2\theta_C+\theta_B)+(FEM)_{CB}+(\frac{2EI}{L})_{CD}(2\theta_C+\theta_D)-(FEM)_{CD}=0 \qquad (k)$$

$$Jo\,\text{int}(D):$$

$$(\frac{2EI}{l})_{CD}(2\theta_D+\theta_C)=0 \qquad (l)$$

- **These equations would involve the four unknown rotations $\theta_A, \theta_B, \theta_C, \theta_D$.**
- **Solving for these four unknown rotations. It may be noted that there is no relative deflection between the supports, so that $\psi = 0$. The values of the obtained rotations may then be substituted in to the slope deflection equations to determine the internal moments at the ends of each member.**
- **If any of the results are negative, they indicate counterclockwise rotation.**

Example (1)

Solution

Draw the shear and moment diagrams for the beam shown in Fig.(3). EI is constant.

1. **Using the formulas for the (FEM) tabulated in Table (1) for the given loadings:**

$$Span\ AB : (FEM)_{AB} = -4.5kN,\ (FEM)_{BA} = +4.5kN$$

$$Span\ BC : (FEM)_{BC} = -1.62kN, (FEM)_{CB} = +1.62$$

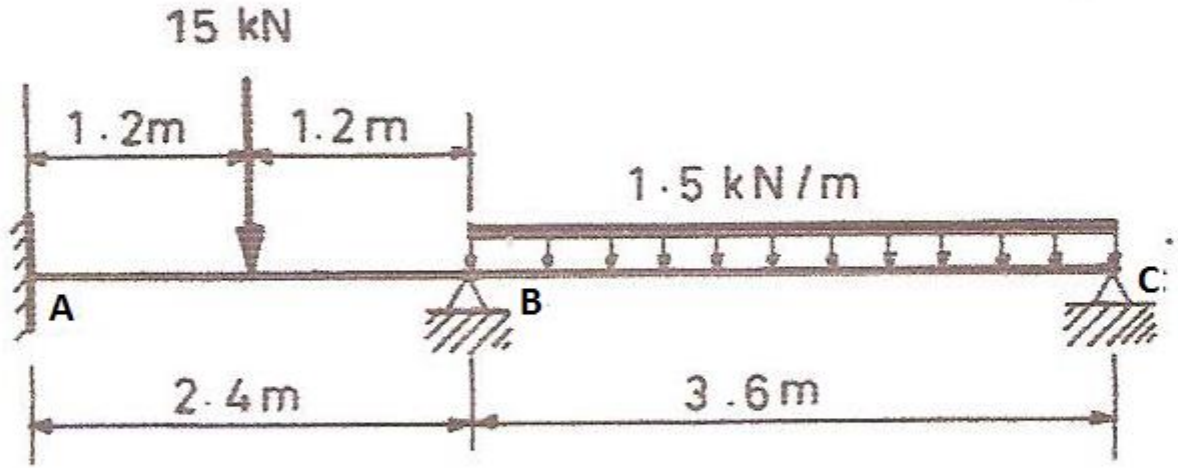


Fig. (3)

2. There are two slopes at B and C, i., e., θ_B, θ_C are unknowns. Since end A is fixed, $\theta_A = 0$. Also, since the supports do not settle, nor are they displaced up or down $\psi_{AB} = \psi_{BC} = 0$.

$$M_{AB} = \left(\frac{2EI}{2.4}\right)_{AB}(\theta_B) - 4.5 \qquad (a)$$

$$M_{BA} = \left(\frac{2EI}{2.4}\right)_{AB}(2\theta_B) + 4.5 \qquad (b)$$

$$M_{BC} = \left(\frac{2EI}{3.6}\right)_{BC}(2\theta_B + \theta_C) - 1.62 \qquad (c)$$

$$M_{CB} = \left(\frac{2EI}{3.6}\right)_{BC}(2\theta_C + \theta_B) + 1.62 \qquad (d)$$

Now, by applying the equilibrium conditions:

$$\left(\frac{2EI}{2.4}\right)\ (2\theta_B)+4.5+\left(\frac{2EI}{3.6}\right)\ (2\theta_B+\theta_C)-1.62=0 \qquad (1)$$

$$\left(\frac{2EI}{3.6}\right)\ (2\theta_C+\theta_B)+1.62=0 \qquad (2)$$

By solving equations (1 &2) for θ_B, θ_C :

$$\theta_B=-\frac{0.828}{EI}\ and\ \theta_C=-\frac{1.044}{EI}$$

Substituting the computed values in to moment equations (a), (b), (c), and (d):

$$M_{AB} = -5.19 \quad kN.m$$

$$M_{BA} = +3.12 \quad kN.m$$

$$M_{BC} = -3.12 \quad kN.m$$

$$M_{CB} = 0$$

By considering the values of support moments and the applied loads, the support reactions may then be determined:

RA = 8.3625 kN ↑

RB = 10.2042 kN ↑

RC = 1.8333 kN ↑

Shearing force and bending moment digrams are shown in Fig. (4).

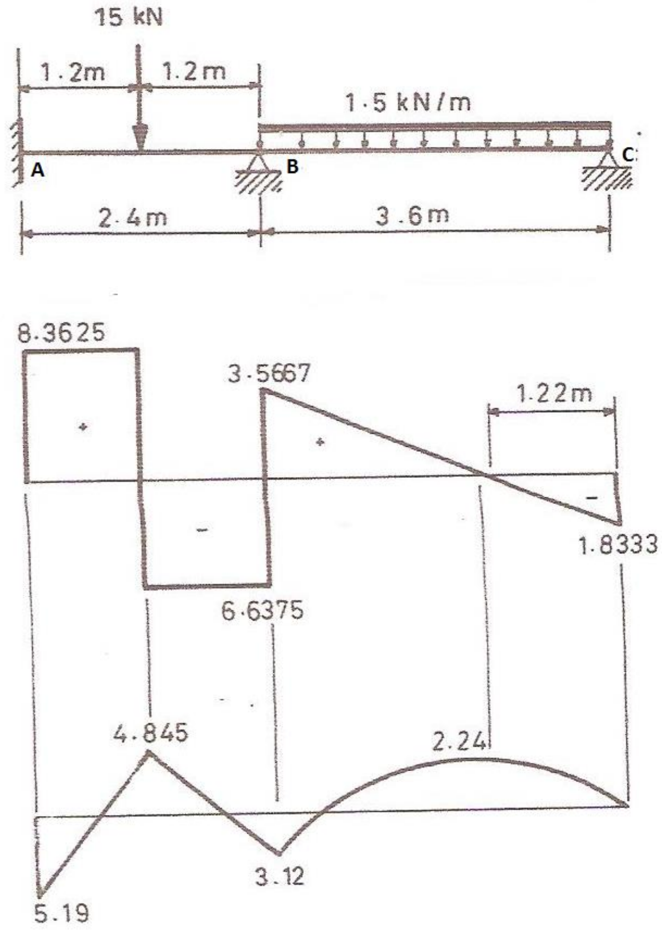
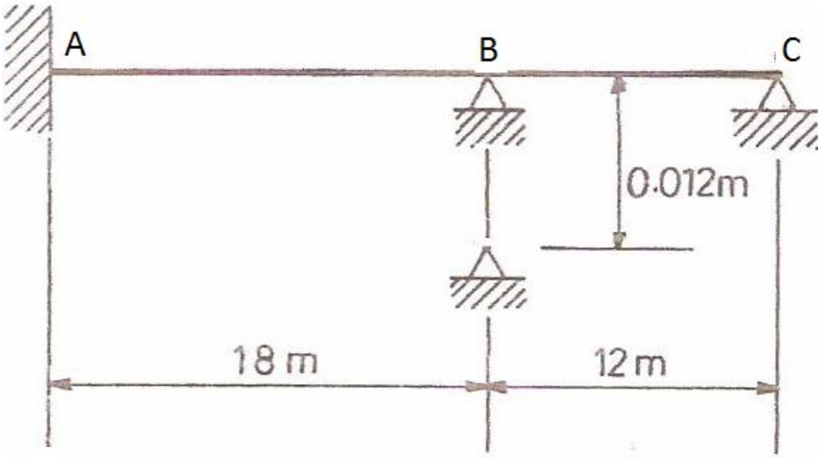


Fig. (4) Shearing Force & Bending Moment Digrams

Example (2)

Determine the internal moments at the supports of the beam shown in Fig. (5). The support at B is displaced (settles) 12 mm.

$E = 200\ kN/mm^2 \quad and \quad I = 1.8225 \times 10^{10}\ mm^4$



Solution

1. Two spans must be considered. FEMs are determined using Table (1).

$$\psi_{AB} = \frac{0.012 - 0}{18} = 6.667 \, x10^{-4}$$

$$\psi_{BC} = \frac{0 - 0.012}{12} = 0.1 x10^{-4}$$

2. Using equation 2:

$$M_{AB} = (\frac{2EI}{18})_{AB} (\theta_B - 3x6.667 \ x10^{-4}) \qquad (i)$$

$$M_{BA} = (\frac{2EI}{18})_{AB} (2\theta_B - 3x6.667 \ x10^{-4}) \qquad (j)$$

$$M_{BC} = (\frac{2EI}{12})_{BC} (2\theta_B + \theta_C + 3x0.1x10^{-4}) \qquad (k)$$

$$M_{CB} = (\frac{2EI}{12})_{BC} (2\theta_C + \theta_B + 3x0.1x10^{-4}) = 0 \qquad (l)$$

3. Equilibrium condition:

$$\sum M_B = 0 \; and \; M_C = 0$$

$$M_{AB} = \left(\frac{2EI}{18}\right)_{AB} (\theta_B - 3x6.667\ x10^{-4}) \qquad (m)$$

$$\left(\frac{2EI}{18}\right)_{AB} (2\theta_B - 3x6.667\ x10^{-4}) + \left(\frac{2EI}{12}\right)_{BC} (2\theta_B + \theta_C + 3x0.1x10^{-4}) = 0 \qquad (n)$$

$$\left(\frac{2EI}{12}\right)_{BC} (2\theta_C + \theta_B + 3x0.1x10^{-4}) = 0 \qquad (p)$$

- In order to obtained the rotations θ_B *and* θ_C equations (n) & (p) may then be solved simultaneously, it may be noted that $\theta_A = 0$ since A is fixed support. Thus,

$$\theta_B = +4.65294 \ x10^{-4} \ rad. \ and \ \theta_C = -2.47647 \ x10^{-4} \ rad.$$

Substituting these values into equations (i to l) yields

$$M_{AB} = -621556 \ kN.mm$$

$$M_{BA} = -433112 \ kN.mm$$

$$M_{BC} = +433112 \ kN.mm$$

$$M_{CB} = 0$$

Example (3)

If end A in example (1) is simply supported, and by applying the compatibility condition, their will be three unknown rotations, $(\theta_A, \theta_B, \theta_C)$

Now,

$$M_{AB} = \left(\frac{2EI}{2.4}\right)_{AB} (2\theta_A + \theta_B) - 4.5 \qquad (a)$$

$$M_{BA} = \left(\frac{2EI}{2.4}\right)_{AB} (2\theta_B) + 4.5 \qquad (b)$$

$$M_{BC} = \left(\frac{2EI}{3.6}\right)_{BC} (2\theta_B + \theta_C) - 1.62 \qquad (c)$$

$$M_{CB} = \left(\frac{2EI}{3.6}\right)_{BC} (2\theta_C + \theta_B) + 1.62 \qquad (d)$$

Applying the equilibrium conditions:

$$M_{AB} = \left(\frac{2EI}{2.4}\right)_{AB}(2\theta_A + \theta_B) - 4.5 = 0 \qquad (1)$$

$$\left(\frac{2EI}{2.4}\right)_{AB}(2\theta_B) + 4.5 + \left(\frac{2EI}{3.6}\right)_{BC}(2\theta_B + \theta_C) - 1.62 = 0 \qquad (2)$$

$$M_{CB} = \left(\frac{2EI}{3.6}\right)_{BC}(2\theta_C + \theta_B) + 1.62 = 0 \qquad (3)$$

By solving equations (1, 2 & 3) for $\theta_A, \theta_B, \theta_C$ and substitute the values into equations (a, b, c, d):

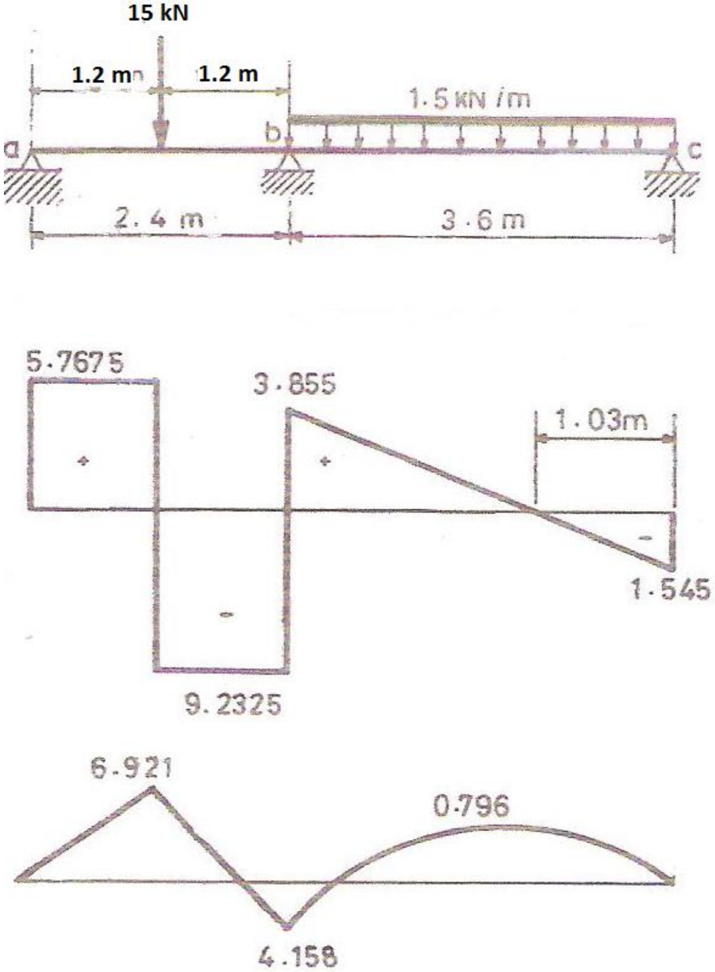
$$M_{AB} = 0$$

$$M_{BA} = +4.158 \ kN.m \quad clockwise$$

$$M_{BC} = -4.158 \ kN.m \quad anticlockwise$$

$$M_{CB} = 0$$

Shearing force and bending moment diagrams are shown in the following



Example (4)

Determine the moments at each joint of the frame shown in Fig.(7). EI is constant.

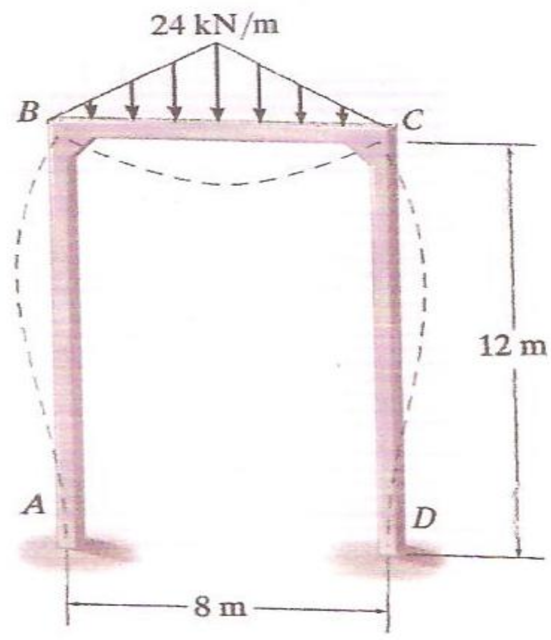


Fig. (7)

$$(FEM)_{BC} = -\frac{5(24)(8)^2}{96} = -80\ kN.M$$

$$(FEM)_{CB} = +\frac{5(24)(8)^2}{96} = +80\ kN.M$$

Because ends A and D are fixed supports. $\theta_A = \theta_D = 0$

and $\psi_{AB} = \psi_{BC} = \psi_{CD} = 0$ since no sidesway will occur.

$$M_{AB} = (\frac{2EI}{12})_{AB}(+\theta_B) = 0.1667\ EI\theta_B$$

$$M_{BA} = (\frac{2EI}{12})_{AB}(+2\theta_B) = 0.3334\ EI\theta_B$$

$$M_{BC} = (\frac{2EI}{8})_{BC}(2\theta_B + \theta_C) - 80 = 0.5EI\theta_B + 0.25EI\theta_C - 80$$

$$M_{CB} = (\frac{2EI}{8})_{BC}(2\theta_C + \theta_B) + 80 = 0.5EI\theta_C + 0.25EI\theta_B + 80$$

$$M_{CD} = (\frac{2EI}{12})_{CD}(2\theta_C) = 0.3334\ EI\theta_C$$

$$M_{DC} = (\frac{2EI}{l})_{CD}(\theta_C) = 0.1667\ EI\theta_C$$

Equilibrium conditions:

$$\sum M_{BA} + M_{BC} = 0$$

$$0.3334\ EI\,\theta_B + 0.5\,EI\,\theta_B + 0.25\,EI\,\theta_C - 80 = 0$$

Or

$$0.8334\ EI\,\theta_B + 0.25\,EI\,\theta_C - 80 = 0 \qquad (1)$$

$$\sum M_{CB} + M_{CD} = 0$$

$$0.5\,EI\,\theta_C + 0.25\,EI\,\theta_B + 0.3334\ EI\,\theta_C + 80 = 0$$

Or

$$0.8334\ EI\,\theta_C + 0.25\,\theta_B + 80 = 0 \qquad (2)$$

Solving simultaneously yields

$$\theta_B = \frac{137.1}{EI} \quad and \quad \theta_C = -\frac{137.1}{EI}$$

Therefore,

$$M_{AB} = 22.9 \ kN.m \quad clockwise$$

$$M_{BA} = 45.7 \ kN.m \quad clockwise$$

$$M_{BC} = -45.7 \ kN.m \quad anticlockwise$$

$$M_{CB} = 45.7 \ kN.m \quad clockwise$$

$$M_{CD} = -45.7 \ kN.m \quad anticlockwise$$

$$M_{DC} = -22.9 \ kN.m \quad anticlokwise$$

The bending moment diagram is shown in Fig.(8).

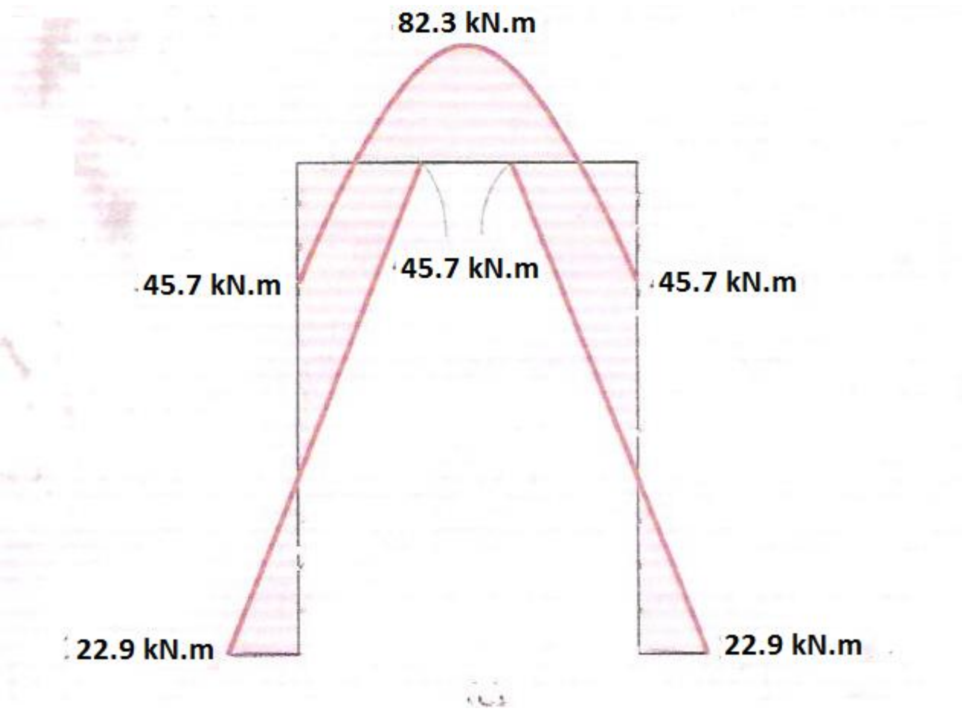


Fig.(8)

Example (5)

Determine the internal moments at each of the frame shown in Fig.(9).

Solution

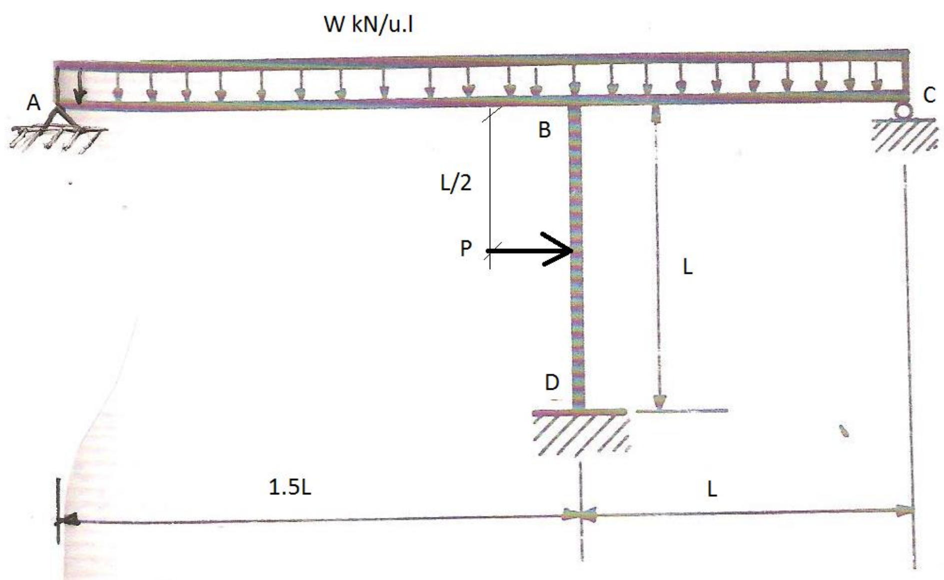


Fig.(9)

1. Fixed end moments:

$$Span \quad AB : (FEM)_{AB} = -\frac{w(1.5L)^2}{12}, \quad (FEM)_{BA} = +\frac{w(1.5L)^2}{12}$$

$$Span \quad BC : (FEM)_{BC} = -\frac{wL^2}{12}, (FEM)_{CB} = +\frac{wL^2}{12}$$

$$Span \quad BD : (FEM)_{DB} = -\frac{Pl}{8}, \quad (FEM)_{BD} = +\frac{Pl}{8}$$

2. Joint moments:

$$M_{AB} = \left(\frac{2EI}{1.5L}\right)_{AB}(2\theta_A + \theta_B) - \frac{w(1.5L)^2}{12} \qquad (i)$$

$$M_{BA} = \left(\frac{2EI}{1.5L}\right)_{AB}(2\theta_B + \theta_A) + \frac{w(1.5L)^2}{12} \qquad (j)$$

$$M_{BC} = \left(\frac{2EI}{L}\right)_{BC}(2\theta_B + \theta_C) - \frac{wL^2}{12} \qquad (k)$$

$$M_{CB} = \left(\frac{2EI}{L}\right)_{BC}(2\theta_C + \theta_B) + \frac{wL^2}{12} \qquad (l)$$

$$M_{BD} = \left(\frac{2EI}{L}\right)_{BD}(2\theta_B + \theta_D) + \frac{PL}{8} \qquad (m)$$

$$M_{DB} = \left(\frac{2EI}{L}\right)_{BD}(2\theta_D + \theta_B) - \frac{PL}{8} \qquad (n)$$

3. Equilibrium conditions:

$$M_{AB} = (\frac{2EI}{1.5L})_{AB}(2\theta_A + \theta_B) - \frac{w(1.5L)^2}{12} = 0 \qquad (1)$$

$$\sum M_B = 0$$

$$(\frac{2EI}{1.5L})_{AB}(2\theta_B + \theta_A) + \frac{w(1.5L)^2}{12} + (\frac{2EI}{L})_{BC}(2\theta_B + \theta_C) - \frac{wL^2}{12} +$$

$$(\frac{2EI}{L})_{BD}(2\theta_B + \theta_D) + \frac{PL}{8} = 0 \qquad \vdots \qquad (2)$$

$$M_{CB} = (\frac{2EI}{L})_{BC}(2\theta_C + \theta_B) + \frac{wL^2}{12} = 0 \qquad (3)$$

$$M_{DC} = (\frac{2EI}{L})_{BD}(\theta_B) - \frac{PL}{8} \qquad (4)$$

Solving equations (1,2,3) simultaneously yields $\theta_A, \theta_B, \theta_C$

Substituting the rotation values into equations (i to n) to determine the joint moments.

Example (6)

Determine the joint internal moments of the frame shown in Fig.(10), both ends A and D are fixed.

Assume $(\frac{EI}{L})_{AB} = (\frac{EI}{L})_{BC} = 1 \; and \; (\frac{EI}{L})_{CD} = 1.5$

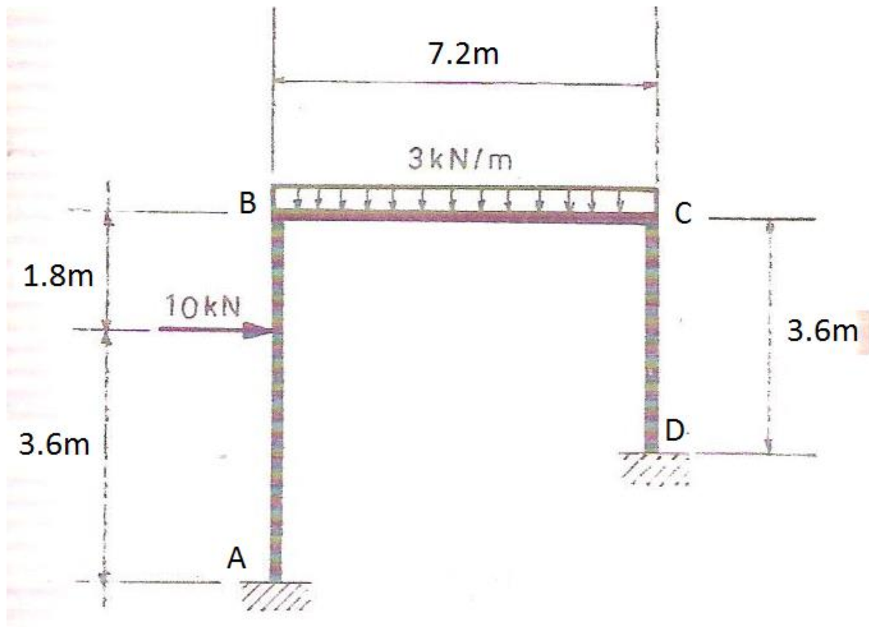


Fig.(10)

Solution

1. Fixed end moments:

$$Span\ AB:\ (FEM)_{AB} = -\frac{10(3.6)(1.8)^2}{(5.4)^2} = -4.0\ kN.m$$

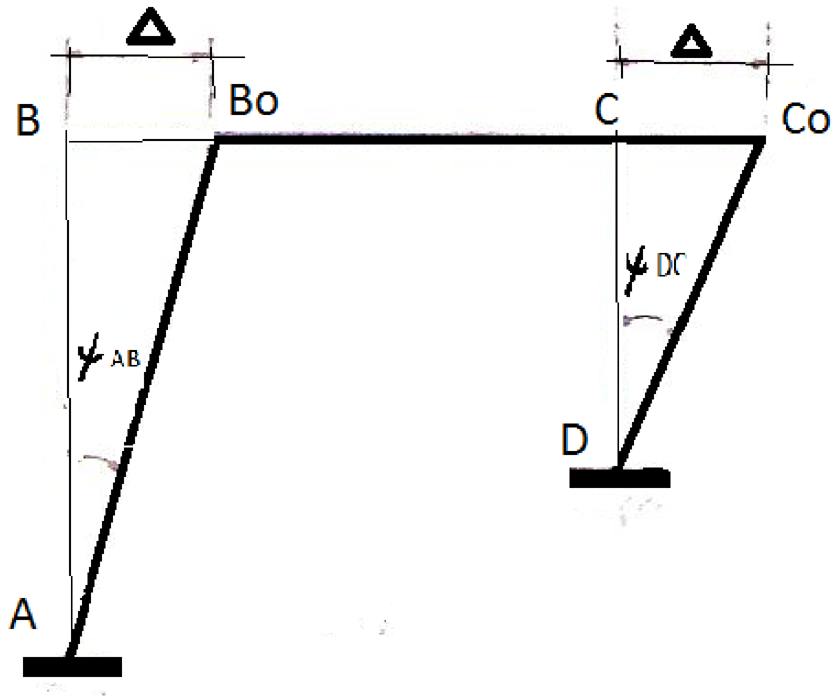
$$(FEM)_{BA} = +\frac{10(1.8)(3.6)^2}{(5.4)^2} = 8.0\ kN.m$$

$$Span\ BC:\ (FEM)_{BC} = -\frac{3(7.2)^2}{12} = -12.96\ kN.m$$

$$(FEM)_{CB} = +\frac{3(7.2)^2}{12} = +12.96\ kN.m$$

It is assumed that the axial deformation is neglected so that

$BB_O = CC_O = \Delta$ as shown in the following figure.



It may be noted that $\psi_{CD} = 1.5\psi_{AB}$ *and* $\psi_{BC} = \theta_A = \theta_D = 0$

2. Joint moments:

$$M_{AB} = (\theta_B - 3\psi_{AB}) - 4 \qquad (i)$$

$$M_{BA} = \left(\frac{2EI}{1.5L}\right)_{AB} (2\theta_B - \psi_{AB}) + 8 \qquad (j)$$

$$M_{BC} = (2\theta_B + \theta_C) - 12.96 \qquad (k)$$

$$M_{CB} = 1.5(2\theta_C + \theta_B) + 12.96 \qquad (l)$$

$$M_{CD} = (2\theta_C - 4.5\psi_{AB})$$

$$= 3\theta_C - 6.75\psi_{AB} \qquad (m)$$

$$M_{DC} = 1.5(\theta_C - 4.5\psi_{AB})$$

$$= 1.5\theta_C - 6.75\psi_{AB} \qquad (n)$$

3. Equilibrium conditions:

$$Joint\ B : \sum M_{BA} + M_{BC} = 0$$

Or

$$4\theta_B + \theta_C - 3\psi_{AB} = 4.96 \qquad (1)$$

$$Joint\ C : \sum M_{CB} + M_{CD} = 0$$

Or

$$\theta_B + 5\theta_C - 6.75\psi_{AB} = -12.96 \qquad (2)$$

Since a horizontal displacement Δ occurs, the summing forces on the entire frame in the x-direction. This yields

$$\rightarrow +\sum F_X = 0: \quad +10 - H_A - H_D = 0$$

In which :

$$H_A = \frac{10}{3} + \frac{M_{AB} + M_{BA}}{5.4}$$

and

$$H_D = \frac{M_{CD} + M_{DC}}{3.6}$$

$$\therefore \quad +10 - \frac{10}{3} + \frac{M_{AB} + M_{BA}}{5.4} + \frac{M_{CD} + M_{DC}}{3.6} = 0$$

Or

$$\theta_B + 2.25\theta_C + 4.75\psi_{AB} = 10.667 \qquad (3)$$

Solving equations (1, 2, 3) yields

$$\theta_B = +2.8208 \ , \theta_C = -0.565 \ , \psi_{AB} = +1.9194$$

By substituting these values into moment equations (i to n):

$$M_{AB} = -6.9374 \ kN.m$$

$$M_{BA} = +7.8833 \ kN.m$$

$$M_{BC} = -7.8833 \ kN.m$$

$$M_{CB} = +14.6509 \ k.N.m$$

$$M_{CD} = -14.6509 \ kN.m$$

$$M_{DC} = -13.8035 \ kN.m$$

B