NUMERICAL PROBLEMS ON ANALYSIS OF THREE HINGED ARCH

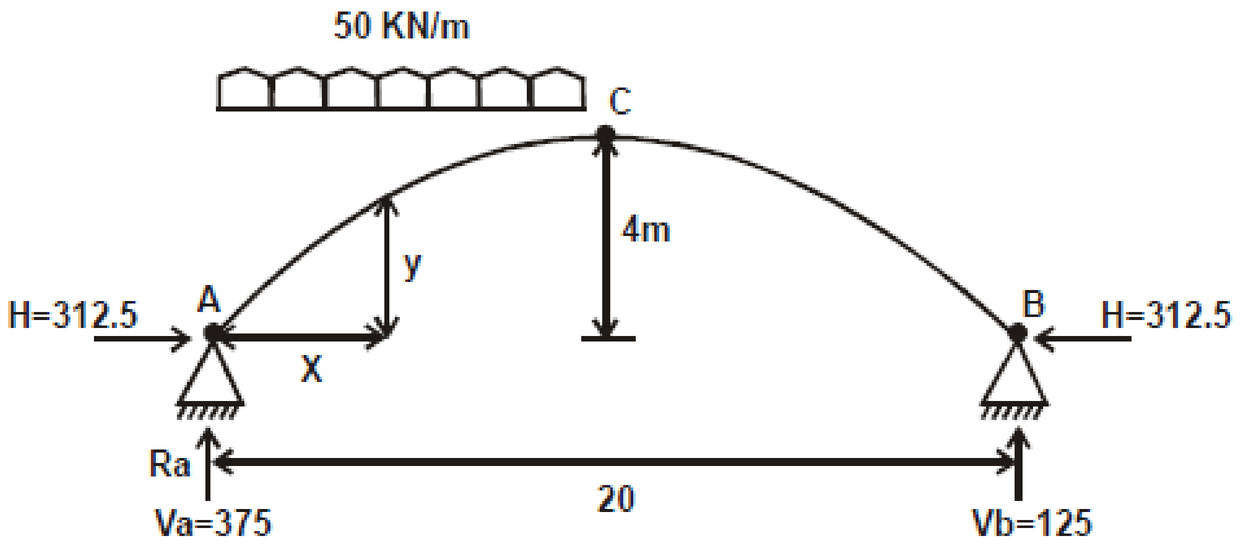
EXAMPLE NO. 1

Analyze a three-hinged arch of span 20m and a central rise of 4m. It is loaded by udl of 50 KN/m over its left half. Calculate maximum positive and negative moments if

(i) The arch is parabolic

(ii) The arch is circular

SOLUTION: 1. Arch is Parabolic



$\sum Ma = 0$

$V_b \times 20 = 50 \times 10 \times 5$

$$V_b = \frac{2500}{20} = 125 \text{ KN}$$

$V_a + V_b = 50 \times 10 = 500 \text{ KN}$

So $V_a = 500 - V_b = 500 - 125$

$= 375 \text{ KN}$

$$H = \frac{\mu_c}{y_c} = \frac{125 \times 10}{4} = 312.5 \text{ KN}$$

$$H = 312.5 \text{ KN}$$

$$R_a = \sqrt{Va^2 + H^2}$$

$$= \sqrt{375^2 + 312.5^2} = \sqrt{140625 + 9765.25}$$

$$= \sqrt{238281.25} = 488.14 \text{ KN}$$

$$\text{Tan}\theta a = \frac{Va}{H} = \frac{375}{312.5} = 1.2$$

$$\theta_a = 50.19^o$$

and $R_b = \sqrt{Vb^2 + H^2} = \sqrt{125^2 + 312.5^2}$

$$Rb = \sqrt{15625 + 97656.25}$$

$$Rb = \sqrt{113281.25} = 336.57 \text{ KN}$$

$$\text{Tan}\theta b = \frac{Vb}{H} = \frac{125}{312.5} = 0.4$$

$$\theta_b = 21.80^0$$

Maximum positive Moment

It is expected in portion AC. Write generalize M_x expression.

$$M_x = 375X - \frac{50X^2}{2} - 312.5y$$

$$\text{Now } y = \frac{4y_c}{L^2}(L - X) = \frac{4 \times 4}{20^2} X(20 - X) = 0.04\,(20X - X^2)$$

$$y = 0.8 - 0.04X^2$$

So

$$M_x = 375X - 25X^2 - 312.5\,[0.8X - 0.04X^2]$$

$$= 375X - 25X^2 - 250X + 12.5X^2 \qquad \text{Simplifying}$$

$$M_x = 125X - 12.5X^2$$

$$\frac{dM_x}{dX} = V_x = 0 = 125 - 25X$$

$X = 5$m from A. Putting Value of X in M_x expression above.

So

$$M_{max} = 125 \times 5 - 12.5 \times 5^2$$

$$= 625 - 312.5$$

$$M_{max} = 312.5 \text{ KN-m}$$

Maximum negative moment:

It would occur in portion BC at a distance x from B.

$M_x = 125X - 312.5y$, Putting equation of y.

$$= 125X - 312.5\,(0.8X - 0.04X^2)$$

$$M_x = 125X - 250X + 12.5X^2$$

$$M_x = -125X + 12.5X^2$$

$$\frac{dM_x}{dX} = V_x = 0 = -125 + 25X$$

$X = 5_m$ from B.

So putting value of X in M_x expression above.

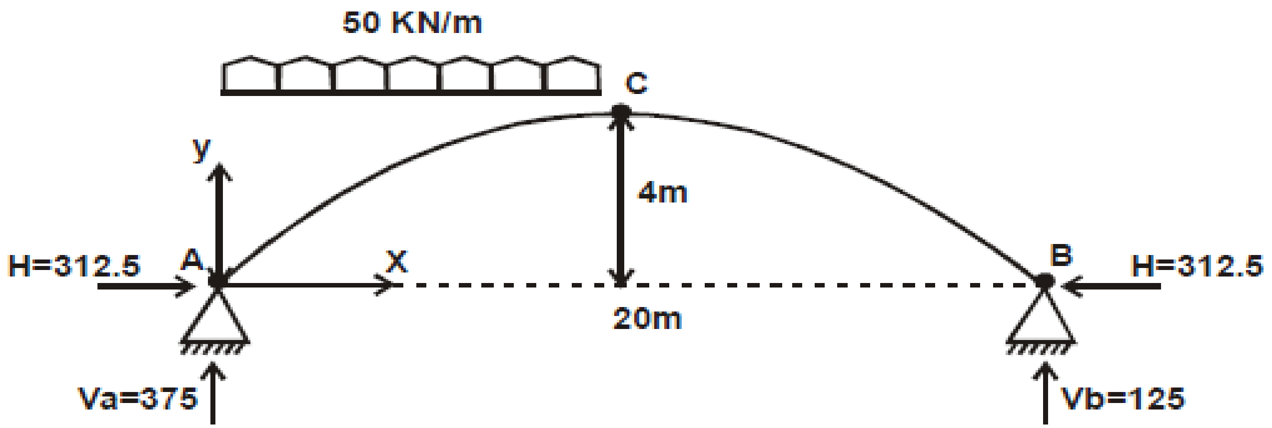
$$M_{max} = -125 \times 5 + 12.5(5)^2$$

$$= -625 + 312.5$$

$$M_{max} = -312.5 \text{ KN-m}$$

SOLUTION: Considering Circular Arch

EXAMPLE NO.2: Now or Solve the following loaded three hinged Circular Arch



Step 1. Reactions:

As before reactions are same.

Step 2. Equation of Circular Arch

The general equation is $(X - h)^2 + (y - k)^2 = r^2$

h and k are co-ordinates at the centre and r is radius of Circle. There are three unknown in above equation, Viz, h, k and r and these can be determined from the following boundary conditions. Origin is at point A.

Boundary conditions

1. At $X = 0$, $y = 0$ It gives $(-h)^2 + (-k)^2 = r^2$

   $h^2 + k^2 = r^2$ (1)

2. At $X = 20$, $y = 0$ It gives $(20 - h)^2 + (-k^2) = r^2$

   $400 + h^2 - 40h + k^2 = r^2$ (2)

3. At $X = 10$, $Y = 4$ It gives $(10 - h)^2 + (4 - k)^2 = r^2$

   $100 + h^2 - 20h + 16 + k^2 - 8k = r^2$

   $116 + h^2 - 20h + k^2 - 8k = r^2$ (3)

Subtract (1) from (2) we get

$400 - 40h = 0$

Or $h = 10$

Put value of h in (1) and 3

$$100 + k^2 = r^2 \qquad (1)$$

$$116 + 100 - 200 + k^2 - 8k = r^2 \qquad (3)$$

or $16 + k^2 - 8k = r^2$ (3)

$16 + k^2 - 8k = 100 + k^2$ (by putting Value of r^2 from 1)

$8k = 16 - 100 = -84$

$$k = \frac{-84}{8} = -10.5$$

Putting $k = -10.5$ in (3)we get

$r^2 = 16 + (-10.5)^2 + 8 \times 10.5$

$= 16 + 110.25 + 84 = 210.25$

So $r = 14.5$ meters.

Putting Values of h, k and r in general equation, we get

$(X - 10)^2 + (y + 10.5)^2 = 14.5^2$ Simplify it, we get.

$$y = -10.5 + \sqrt{14.5^2 - (X - 10)^2}$$

$$(y + 10.5)^2 = 14.5^2 - (X - 10)^2$$

$$= -10.5 + \sqrt{210.25 - X^2 - 100 + 20X}$$

$$y = -10.5 + \sqrt{110.25 - X^2 + 20X} \qquad (4)$$

We know, $yc\,(2r - yc) = \frac{L^2}{4}$ (5)

and

$$y = \sqrt{r^2 - \left(\frac{L}{2} - X\right)^2} - (r - yc) \qquad (6)$$

These equations are same as were used in derivation earlier.

Alternatively to avoid evaluation of constants each time, equations (5) and (6) can be used.

Equation (6) is the equation of Centre-line of Circular arch.

Step 3: Calculation of Maximum moment.

Maximum positive moment occurs in span AC. Write M_x expression

$$M_x = 375X - \frac{50X^2}{2} - 312.5\,y$$ put y from (4) above.

$$= 375X - 25X^2 - 312.5\left[-10.5 + \sqrt{110.25 - X^2 + 20X}\right]$$

$$M_x = 375X - 25X^2 + 3281.25 - 312.5\sqrt{110.25 - X^2 + 20X}$$

Now maximum moment occurs where shear force is zero. So

$$\frac{dM_x}{dx} = V_x = 375 - 50X - \frac{312.5\,(-2X + 20)}{2\sqrt{110.25 - X^2 + 20X}} = 0$$

$$375 - 50X = \frac{312.5\,(-X + 10)}{\sqrt{110.25 - X^2 + 20X}}$$ divide by 50

$$7.5 - X = \frac{6.25\,(10 - X)}{\sqrt{110.25 - X^2 + 20X}}$$ multiply by -1, We get

$$X - 7.5 = \frac{6.25\,(X - 10)}{\sqrt{110.25 - X^2 + 20X}}$$

$$(X - 7.5)\sqrt{110.25 - X^2 + 20X} = 6.25\,(X - 10)$$ square both sides

$$(X - 7.5)^2\,(110.25 - X^2 + 20X) = 6.25^2\,(X - 10)^2$$ Simplify

$$(X^2 - 15X + 56.25)\,(110.25 - X^2 + 20X) = 39.0625\,(X^2 - 20X + 100)$$

or $$110.25X^2 - X^4 + 20X^3 - 1653.75X + 15X^3 - 300X^2 = 39.0625X^2 - 781.25X + 3906.25 + 6201.56 - 56.25X^2 + 1125X$$

Simplifying

$$-X^4 + 35X^3 - 285.0625X^2 + 252.5X + 2295.3125 = 0$$

or $$X^4 - 35X^3 + 285.0625X^2 - 252.5X - 2295.3125 = 0$$

Now it is considered appropriate to solve this equation by Modified Newton – Raphson iteration solutions which in general is

$$X_n + 1 = X_n + \frac{f(X_n)}{f'(X_n)} \qquad (A)$$

$$\text{So } f(X) = X^4 - 35X^3 + 285.0625X^2 - 252.5X - 2295.3125$$

And differentiate, $f'(X) = 4X^3 - 105X^2 + 570.125X - 252.5$

In general, it is recommended that first root X_n should be always taken at 1 because it converges very fast. However, knowing that B. M will be maximum near the middle of portion AC, we take $X_n = 2$ (to reduce number of iterations possibly) and solve in the following tabular form. Evaluate f(X) and $f'(X_n)$ expressions.

Iteration Number	X_n	$f(X_n)$	$f'(X_n)$	$X_n + 1$ from A above
1	2	–1924.06	499.75	5.85
2	5.85	147.251	290.1629	5.3425
3	5.3425	– 30.3142	406.3845	5.417
4	5.417	– 0.58794	390.546	5.418

So we get X_n and $X_n + 1$ as same after 4th iteration.

So $X = 5.418$ m put this in MX expressions

$$M_{max} = 375\,(5.418) - 25\,(5.418)^2 + 3281.25 - 312.5\sqrt{110.25 - 5.418^2 + 20 \times 5.418}$$

$$= 280.066 \text{ KN-m}$$

Maximum negative moment in the arch

Let us assume that it occurs in portion BC at a distance X from A $(10 < X < 20)$

$$M_x = 125\,(20 - X) - 312.5\left(-10.5 + \sqrt{110.25 - X^2 + 20X}\right) \quad \text{Simplify}$$

$$= 2500 - 125X + 3281.25 - 312.5\sqrt{110.25 - X^2 + 20X}$$

or $$M_x = 5781.25 - 125X - 312.5\sqrt{110.25 - X^2 + 20X}$$

Maximum moment occurs where SF is zero, So differentiate M_x expression w.r.t. X.

$$\frac{dM_x}{dX} = 0 = -125 - \frac{312.5\,(-2X + 20)}{2\sqrt{110.25 - X^2 + 20X}}$$

or $$0 = -125\ \frac{+\ 312.5(X - 10)}{\sqrt{110.25 - X^2 + 20X}}$$

$125\sqrt{110.25 - X^2 + 20X} = 312.5\,(X - 10)$ squaring both sides. We have,

$15625\,(110.25 - X^2 + 20X) = 97656.25\,(X^2 - 20X + 100)$ Simplify

$$110.25 - X^2 + 20X = 6.25\,(X^2 - 20X + 100)$$

$$0 = 7.25X^2 - 145X + 514.75 \quad \text{dividing by 7.25}$$

$X^2 - 20X + 71 = 0$ Solve this quadretic equation.

$$X = \frac{20 \pm \sqrt{400 - 284}}{2}$$

$$X = \frac{20 \pm 10.77}{2} = 15.385\text{m from A}$$

Put this value of X in M_x expression above.

$$\text{So } M_{max} = 5781.25 - 125 \times 15.385 - 312.5\sqrt{110.25 - (15.385)^2 + 20\,(15.385)}$$

$$= 5781.25 - 1923.125 - 312.5\sqrt{181.257} = -349.115\text{KN.m}$$

THANK YOU